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# Torelli theorem for certain rational surfaces and root system of type A(Combinatorial Aspects in Representation Theory and Geometry)

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# Torelli theorem for certain rational surfaces and root system of type A

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For an integer  $n \geq 2$ , let  $\Sigma_n$  be the  $n$ -th Hirzebruch surface defined by

$$(0.1) \quad \{(\zeta_0 : \zeta_1 : \zeta_2)(s : t) \in \mathbf{P}^2 \times \mathbf{P}^1 \mid s^n \zeta_0 = t^n \zeta_1\},$$

where  $\mathbf{P}^k$  is  $n$ -dimensional complex projective space. Let  $X_n$  be a surface obtained by blowing up  $n + 1$  points of  $\Sigma_n$  and  $D$  be an anti-canonical divisor on  $X_n$  such that  $D$  consists of four nonsingular rational curves and its intersection diagram is a circle (thus  $D$  forms a square).

We study the isomorphism classes of the pairs  $(X_n, D)$ . The isomorphism classes can be characterized in terms of the root system of type A. E. Looijenga investigated the isomorphism classes of rational surfaces with anti-canonical divisors [L]. We deal with another class of rational surfaces. The method and formulation are very similar to those of Looijenga's.

## 1. HIRZEBRUCH SURFACES

We assume  $n \geq 3$ .  $\Sigma_n$  is a subvariety of  $\mathbf{P}^2 \times \mathbf{P}^1$  (cf (0.1)). Let  $\pi : \Sigma_n \rightarrow \mathbf{P}^1$  be the second projection.  $\Sigma_n$  is a  $\mathbf{P}^1$ -bundle over  $\mathbf{P}^1$ . Let  $F$  be a fiber of the projection  $\pi : \Sigma_n \rightarrow \mathbf{P}^1$  and  $S$  be the section defined by  $\zeta_0 = \zeta_1 = 0$ .

DEFINITION. we say that  $n + 1$  points  $P_1, \dots, P_{n+1}$  of  $\Sigma_n$  are in 'general position' if they satisfy the following conditions: (1)  $P_i \neq P_j$  for  $i \neq j$  and (2) there exists a nonsingular curve in the complete linear system  $|nF + S|$  passing through  $P_1, \dots, P_{n+1}$ .

REMARK. If  $P_1, \dots, P_{n+1}$  are in general position, then  $P_i \notin S$  and no two of  $P_i$  are on a fiber.

Let  $p : X_n \rightarrow \Sigma_n$  be the morphism obtained by blowing up  $n + 1$  points  $P_1, \dots, P_{n+1}$  in general position.

LEMMA 1.1. If  $D$  is an anti-canonical divisor on  $X_n$  and satisfies the following conditions:

- (1)  $D$  is the strict transform of an anti-canonical divisor  $D'$  on  $\Sigma_n$ ,

- (2)  $D'$  consists of four irreducible components and its intersection diagram is a circle,  
 (3)  $P_1, \dots, P_{n+1}$  are on only one component of  $D'$  and not on other components,

then

$$D = F_1 + F_2 + S + C,$$

where  $F_i$  is a strict transform of a fiber of the projection  $\pi : \Sigma_n \rightarrow \mathbf{P}^1$ ,  $S$  is the strict transform of the  $(-n)$ -section of  $\Sigma_n$  and  $C$  is the strict transform of the unique nonsingular curve of  $|nF + S|$  passing through  $P_1, \dots, P_{n+1}$ .

NOTATION. We say that an anti-canonical divisor  $D$  on  $X_n$  is of ' $\#$ -type' if it satisfies the condition of lemma 1.1. We denote by  $F_0$  and  $F_\infty$  the components of  $D$  which are the strict transforms of the fibers of  $\pi$ .

## 2. HOMOLOGY AND ROOT SYSTEM

Let  $X_n$  and  $D = F_0 + F_\infty + S + C$  be as in §1. Consider the homology exact sequence:

$$\begin{array}{ccccccc} \dots & & \rightarrow & H_3(X_n; \mathbb{Z}) & \rightarrow & H_3(X_n, X_n - D; \mathbb{Z}) & \\ & & & \parallel & & & \\ & & & 0 & & & \\ \partial_* & H_2(X_n - D; \mathbb{Z}) & \xrightarrow{i_*} & H_2(X_n; \mathbb{Z}) & \xrightarrow{j_*} & H_2(X_n, X_n - D; \mathbb{Z}) & \\ \rightarrow & \dots & & & & & \end{array}$$

We extend the intersection form in  $H_2(X_n; \mathbb{Z})$  to  $H_2(X_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ . Let

$$Q = \ker j_* \subset H_2(X_n; \mathbb{Z})$$

and

$$R = \{\alpha \in Q \mid \alpha \cdot \alpha = -2\}.$$

LEMMA 2.1.  $R$  is a root system of type  $A_n$  in  $Q \otimes_{\mathbb{Z}} \mathbb{R}$  and  $Q$  is generated by  $R$ . The set  $\{e_i - e_{i-1} \mid 1 \leq i \leq n\}$  is the basis of  $R$ , where  $e_i$  is the class of the exceptional curve  $E_i = p^{-1}(P_i)$ .

We now have the short exact sequence:

$$(2.1) \quad 0 \rightarrow H_3(X_n, X_n - D; \mathbb{Z}) \xrightarrow{\partial_*} H_2(X_n - D; \mathbb{Z}) \xrightarrow{i_*} Q \rightarrow 0.$$

LEMMA 2.2 (K. IRIE).

$$H_3(X_n, X_n - D; \mathbb{Z}) \simeq \mathbb{Z}$$

Let  $\varepsilon$  be the generator of  $H_3(X_n, X_n - D; \mathbb{Z})$ . We next consider a meromorphic 2-form on  $X_n$  which has poles only along  $D$ .

**LEMMA 2.3.** *There exists a unique meromorphic 2-form  $\omega$  on  $X_n$  such that*

- (1)  $\omega$  has poles only along  $D$ ,
- (2)  $\omega(\partial_*(\varepsilon)) = 1$ .

Furthermore, we can choose an affine coordinate  $z$  on  $C(\subset D)$  such that  $F_0 \cap C = 0$ ,  $F_\infty \cap C = \infty$  and

$$\text{Res}_C \omega = \frac{1}{(2\pi i)^2} \frac{dz}{z}.$$

It follows from this lemma, we can define a character  $\chi : Q \rightarrow \mathbb{C}^*$  by

$$\chi(i_*[\Gamma]) = \exp 2\pi i \int_\Gamma \omega,$$

where  $\Gamma \in H_2(X_n - D; \mathbb{Z})$ .

$$\begin{array}{ccc} H_2(X_n - D; \mathbb{Z}) & \xrightarrow{\exp 2\pi i \int_{[\cdot]} \omega} & \mathbb{C}^* \\ & \searrow i_* & \uparrow \chi \\ & & Q \end{array}$$

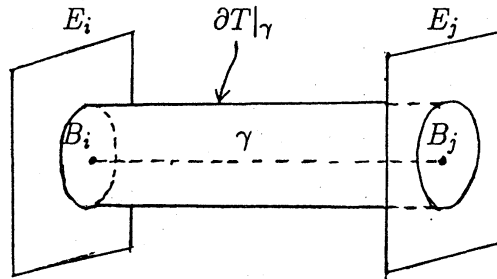
### 3. TORELLI THEOREM FOR THE PAIR $(X_n, D)$

We first consider the value of  $\chi$  at the class  $e_i - e_j \in Q$ , where  $e_i$  and  $e_j$  are the homology classes of the exceptional curves  $E_i = p^{-1}(P_i)$  and  $E_j = p^{-1}(P_j)$  respectively. Let  $B_i = E_i \cap C$  and let  $T$  be a closed tubular neighborhood of  $C$  in  $X_n$  such that  $T \cap E_i$  and  $T \cap E_j$  are fibers. Let  $\gamma$  be an injective path in  $C$  from  $B_i$  to  $B_j$  and let

$$\Gamma_{i,j} = (E_i \setminus (E_i \cap T)) \cup \partial T|_\gamma \cup (E_j \setminus (E_j \cap T)).$$

We can take the orientation such that  $\Gamma_{i,j}$  is homologous to  $E_i - E_j$  in  $X_n$ . Hence we have

$$i_*([\Gamma_{i,j}]) = e_i - e_j.$$



Since  $E_i$  and  $E_j$  are the inverse image of the points  $P_i$  and  $P_j$  respectively, we have

$$\int_{E_i \setminus (E_i \cap T)} \omega = \int_{E_j \setminus (E_j \cap T)} \omega = 0.$$

Therefore

$$\int_{\Gamma_{i,j}} \omega = \int_{\partial T|_\gamma} \omega.$$

By the residue formula, we have

$$\begin{aligned} \int_{\partial T|_\gamma} \omega &= 2\pi i \int_\gamma \text{Res}_C \omega \\ &= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{t_i}^{t_j} \frac{dz}{z} \\ &= \frac{1}{2\pi i} \log \frac{t_j}{t_i} \pmod{\mathbb{Z}}, \end{aligned}$$

where  $t_i$  and  $t_j$  are the affine coordinates of the points  $B_i$  and  $B_j$  respectively. Then we now have

$$\begin{aligned} \chi(e_i - e_j) &= \exp 2\pi i \int_{\Gamma_{i,j}} \omega \\ &= \frac{t_j}{t_i} \end{aligned}$$

The important point is that this is the cross ratio of  $C \cap F_0, C \cap F_\infty, B_j$  and  $B_i$ . Thus we have the theorem of Torelli type.

**THEOREM.** Let  $X_n$  and  $X'_n$  be the surfaces defined in §1 and let  $D$  and  $D'$  be anti-canonical divisors of  $\#$ -type on  $X_n$  and  $X'_n$  respectively (cf. notation in §1). Let denote root lattices by  $Q$  and  $Q'$ , root systems by  $R$  and  $R'$ , and characters by  $\chi$  and  $\chi'$  defined as in §2 for  $X_n$  and  $X'_n$  respectively. If  $\varphi : H_2(X_n; \mathbb{Z}) \rightarrow H_2(X'_n; \mathbb{Z})$  is an isometry such that

- (1)  $\varphi([F_i]) = [F'_i]$ ,
- (2)  $\varphi([C]) = [C']$ ,
- (3)  $\varphi(R) = R'$ ,
- (4)  $\varphi^*(\chi') = \chi$ ,

then there exists a unique isomorphism  $\Phi : X_n \rightarrow X'_n$  which maps  $F_i$  to  $F'_i$  and  $C$  to  $C'$  and induces  $\varphi$ .

## REFERENCE

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